29. Berzin, I. S. and Zhidkov, N. P. Calculation Methods, Moscow, Fizmatgiz, Vol. 2, 1960.

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SPECTRUM OF THE SYSTEM DESCRIBING OSCILLATIONS OF A SHELL OF REVOLUTION

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The relationship between spectra of moment and momentless [membrane] systems of differential equations which describe the characteristic oscillations of shells of revolution is examined.

For the eigenvalues of the lower series the oscillation theorem is proven. Conditions are found for which the lower series of frequencies of the momentless system has a finite limit point.

A number of papers are devoted to finding the frequencies of characteristic oscillations of a thin shell by the small parameter method (see Bibliography).

In this paper some mathematical problems are examined which are connected with the problem of finding the characteristic frequencies for a shell of revolution. In this case the characteristic oscillations with m waves along the parallel are described by the following system of equations [1, 5]:

$$
\begin{gathered}
-u'' - \frac{B'}{B}u' - \frac{m(1+\sigma)}{2B}v' - \left[\left(\frac{B'}{B}\right)' + (1-\sigma)\left(\frac{1}{R_1R_2} - \frac{m^2}{2B^2}\right)\right]u + \\
+ \frac{mB'}{B^2}\frac{3-\sigma}{2}v + \left(\frac{1}{R_1} + \frac{\sigma}{R_2}\right)w' + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)' w = \lambda u \\
-\frac{1-\sigma}{2}v'' + \frac{m}{B}\frac{(1+\sigma)}{2}u' - \frac{1-\sigma}{2}\frac{B'}{B}v' + \frac{mB'}{B^2}\frac{3-\sigma}{2}u - \\
-\left[\frac{1-\sigma}{2}\left(\frac{B'}{B}\right)' + \frac{1-\sigma}{R_1R_2} - \frac{m^2}{B^2}\right]v - \frac{m}{B}\left(\frac{\sigma}{R_1} + \frac{1}{R_2}\right)w = \lambda v \\
\mu^4\frac{1}{B}\left(\frac{d}{ds}B\frac{d}{ds} - \frac{m^2}{B}\right)\frac{1}{B}\left(\frac{d}{ds}B\frac{d}{ds} - \frac{m^2}{B}\right)w - \left(\frac{1}{R_1} + \frac{\sigma}{R_2}\right)\frac{du}{ds} - \\
-\left(\frac{\sigma}{R_1} + \frac{1}{R_2}\right)\frac{B'}{B}u - \frac{m}{B}\left(\frac{\sigma}{R_1} + \frac{1}{R_2}\right)v + \left(\frac{1}{R_1^2} + \frac{2\sigma}{R_1R_2} + \frac{1}{R_2^2}\right)w = \lambda w
\end{gathered}
\qquad (0.1)
$$

Here u, v, w are the projections of the displacement of a point on the directions of the meridian, the parallel and the normal to the shell, respectively; s is the length of the meridian arc, $a \leqslant s \leqslant b$; $B(s)$ is the distance from the meridian to the axis of revolution; $R_1(s)$ and $R_2(s)$ are the principal radii of curvature of the shell

$$
\frac{1}{R_1} = -\frac{B''}{\sqrt{1-B'^2}}, \qquad \frac{1}{R_2} = \frac{\sqrt{1-B'^2}}{B}, \qquad \lambda = (1-\sigma^2)\frac{\gamma}{E}p^2, \quad \mu^4 = \frac{h^2}{12} \qquad (0.2)
$$

where E is Young's modulus, σ is Poisson's ratio, γ is the density, p is the frequency of oscillations, h is the thickness of the shell and μ is the small parameter. The system

(0.1) will be examined for the following boundary conditions:

$$u(a) = u(b) = v(a) = v(b) = w(a) = w(b) = w'(a) = w'(b) = 0 \quad (0.3)$$

This corresponds to rigid fixing of the edges of the shell along two parallels.

If we introduce into the analysis the vector function $f(s) = (u(s), v(s), w(s))$, we write the system (0.1) in the abbreviated form

$$L_\mu f = \lambda f \quad (0.4)$$

For boundary conditions (0.3) the operator L_μ is self-adjoint and positive definite if the scalar product is determined from the following equation:

$$(f_1, f_2) = \int_a^b B(s)(u_1 u_2 + v_1 v_2 + w_1 w_2)\,ds \quad (0.5)$$

The spectrum of the operator L_μ for $\mu \neq 0$ is discrete and its eigenvalues $\lambda_k(\mu)$ are positive. In this connection $\lim \lambda_k(\mu) = \infty$ for $k \to \infty$. The eigenvectors $f_k(s, \mu)$ $(k = 1, 2, \ldots)$ are orthogonal for any arbitrary vector function $g(s)$. The following expansion (*) which converges in metric (0.4) is valid:

$$g(s) = \sum_{k=1}^{\infty} c_k(\mu) f_k(s, \mu) \quad (0.6)$$

Let us set $\mu = 0$ in (0.4). Then a degenerate (momentless) operator L_0 arises and is self-adjoint if in (0.3) the boundary conditions imposed on $w(s)$ are removed. In fact, it is easy to verify that

$$(L_0 f_1, f_2) = (f_1, L_0 f_2)$$

for any f_1 and f_2, which satisfy the conditions

$$u(a) = u(b) = v(a) = v(b) = 0 \quad (0.7)$$

Furthermore, the operator L_0 is positive definite. For any real f, which satisfies the conditions (0.7), we have

$$(L_0 f, f) = \int_a^b B(s)\Big\{\sigma\Big[u' + \frac{B'}{B}u + \frac{m}{B}v + \Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)w\Big]^2 + (1-\sigma)\Big(u' + \frac{w}{R_1}\Big)^2 + (1-\sigma)\Big(\frac{B'}{B}u + \frac{m}{B}v + \frac{w}{R_2}\Big)^2 + \frac{1-\sigma}{2}\Big(-\frac{m}{B}u + v' - \frac{B'}{B}v\Big)^2\Big\}ds \geqslant 0$$

The spectrum of the problem

$$L_0 f = \lambda f \quad (0.8)$$

now will not be purely discrete. In Sect. 2 it is proven that the interval $[\alpha, \beta]$ of values of the function

$$\varphi_1(s) = \frac{1-\sigma^2}{R_2^2(s)} \qquad (a \leqslant s \leqslant b) \quad (0.9)$$

(*) If $g(s)$ is a smooth vector function satisfying conditions (0.3), then the series (0.6) converges uniformly in $s \in [a, b]$.

belongs to the continuous spectrum of the problem (0. 8), (0. 7) [4-6]. Outside the interval α, β the spectrum is discrete. The ends of the interval, α and β can be points of concentration for the eigenvalues of the operator L_0, for small α and large β. Assuming for definiteness that $\lambda = 0$ will not be a point of the spectrum (*) of the operator L_0, it is possible to show (the proof is presented in the Appendix) the strong convergence of operator L_μ^{-1} to L_0^{-1} for $\mu \to 0$. This means that for any vector function $g(s)$

$$\lim_{\mu\to 0} \| (L_\mu^{-1} - L_0^{-1}) g \| = 0, \qquad \|g\| = (g, g)^{1/2} \tag{0.10}$$

Here the scalar product (g, g) is determined according to Eq. (0. 5). On the basis of a theorem in the general theory of perturbations [7] the indicated situation leads to a strong convergence of the spectral function of the operator L_μ to the spectral function of the momentless (degenerate) operator L_0. For details the reader is referred to monograph [7]. However, the conclusion drawn here will be clarified through the following important corollary which is based on this conclusion. Let λ_0 be an isolated eigenvalue of the operator L_0 (for simplicity single) and $f_0(s)$ the corresponding eigenfunction. Let ε be so small that in the interval Δ_ε $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ there are no eigenvalues of L_0, other than λ_0.
Also let

$$g_\mu^*(s) = \sum_{\lambda_k(\mu)} c_k(\mu) f_k(s, \mu) \qquad (\lambda_k(\mu) \in \Delta_\varepsilon)$$

be the interval of expansion of $g(s)$ in the series (0. 6) corresponding to those eigenvalues $\lambda_k(\mu)$, which belong to Δ_ε (for $\mu \to 0$ their number can increase (!)). Furthermore we have always

$$\lim \| g_\mu^*(s) - (g, f_0) f_0(s) \| = 0$$

In particular, for $g(s) = f_0(s)$ the indicated interval of the Fourier series tends in the mean square to $f_0(s)$.

It is clear that the above mentioned fact does not contain complete information on transformation of eigenvalues $\lambda_k(\mu)$ into the spectrum of operator L_0. This problem requires special examination. We note, however, that for eigenvalues of problem (0. 7) (0. 8 which are smaller than $\alpha = \inf \varphi_1(s)$ (see (0. 9)) the conditions of regular degeneracy are satisfied in the sense given by the authors of [8]. For these $\lambda_k(\mu)$ it is therefore possible to write the following asymptotic equations in analogy to the way it was handled in [9] in the axisymmetric case

$$\lambda_k(\mu) = \lambda_k^\circ + \mu\lambda_k' + o(\mu)$$

This paper is devoted to the investigation of the spectrum of the momentless system (0. 7), (0. 8). A large part of theorems given below is a generalization of results obtained earlier for the case of axisymmetric oscillations and presented in [6, 9].

In Sect. 3 the oscillation theorem is proven for the system (0. 7), (0. 8). In Sect. 4 the sufficient conditions are found for which the first series of frequencies (the least) is infinite.

(*) Otherwise L_μ and L_0 should be replaced by $L_\mu + \varkappa$ and $L_0 + \varkappa$ respectively, with $\varkappa > 0$.

1. The Cauchy problem for the momentless system. In (0.1) we set $\mu = 0$. Introducing the vector $y = (u, v)$ we rewrite the first two equations of system (0.1) in the form

$$A_0 y'' + A_1 y' + A_2 y - \lambda A_3 y = A_4 d + A_5 e \tag{1.1}$$

Here d and e are vectors

$$d = (w', 0), \; e = (w, w) \tag{1.2}$$

Through $A_k(s)$ $(k = 0, 1, \ldots, 5)$ we denote matrices of the second order. The elements of these matrices can be easily reconstructed from system (0.1). In particular

$$A_0 = \begin{pmatrix} -1 & 0 \\ 0 & -{}^1/_2(1-\sigma) \end{pmatrix}, \quad A_4 = -\begin{pmatrix} R_1^{-1} + R_2^{-1}\sigma & 0 \\ 0 & 0 \end{pmatrix} \tag{1.3}$$

The third equation of system (0.1) for $\mu = 0$ has the form

$$-b_0 u' - b_1 u - b_2 v + (\varphi_2(s) - \lambda) w = 0 \tag{1.4}$$

Here we have introduced the well-understood designations for coefficients. In particular

$$b_0 = R_1^{-1} + \sigma R_2^{-1}, \qquad \varphi_2(s) = R_1^{-2} + 2\sigma R_1^{-1} R_2^{-1} + R_2^{-2} \tag{1.5}$$

It is easy to verify that $\varphi_2(s) - \varphi_1(s) = (R_1^{-1} + \sigma R_2^{-1})^2 \geqslant 0$. The function $\varphi_1(s)$ is determined by Eq.(0.9). The interval of values $\varphi_1(s)$ is denoted by $[\alpha, \beta]$. The interval of values $\varphi_2(s)$ for $a \leqslant s \leqslant b$ will be $[\gamma, \delta]$. Assuming that the coefficients of the system (0.1) are piecewise-continuous, we prove the following statement.

Lemma 1.1. Let r_1, r_2, r_3, r_4 be arbitrary real numbers and $s_0 \in [a, b]$. Then for $\lambda \overline{\in} [\alpha, \beta]$ and $\lambda \neq \varphi_2(s_0)$ there exists a unique solution $f(s, \lambda) = (u(s, \lambda), v(s, \lambda), w(s, \lambda))$ of the system (0.8) which satisfies Cauchy's conditions(*)

$$u(s_0, \lambda) = r_1, \quad v(s_0, \lambda) = r_2, \quad u'(s_0, \lambda) = r_3, \quad v'(s_0, \lambda) = r_4 \tag{1.6}$$

The vector function $f(s, \lambda)$ is analytic for all complex $\lambda \overline{\in} [\alpha, \beta]$ and $\lambda \neq \varphi_2(s_0)$.

Proof. We express u and v from system (1.1) through w and substitute the result into (1.4). We obtain the Volterra equation with respect to $w(s, \lambda)$, which allows the proof of the lemma. System (1.1) written in the abbreviated form is

$$l_\lambda y = p \tag{1.7}$$

Let

$$Y(s, \lambda) = \begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix} \tag{1.8}$$

be a fundamental matrix of the dimension 4x4 composed of solutions of the system

$$l_\lambda y = 0 \tag{1.9}$$

The matrix also satisfies the condition $Y(s_0, \lambda) = E_4$ (E_4 is a unit matrix). Let

$$Z(s, \lambda) = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} \tag{1.10}$$

be the inverse matrix of $Y(s, \lambda)$. Both matrices (1.8) and (1.10) will be entire functions of λ. It is easy to verify that the solution $y(s, \lambda)$ of the system (1.7) can be represented in the following form:

(*) We recall that system (0.8) is obtained from system (0.1) for $\mu = 0$.

$$y(s,\lambda) = y_0(s,\lambda) + \int_{s_0}^{s} C(s,t) A_0^{-1} p(t)\, dt \tag{1.11}$$

The solution satisfies conditions (1.6). In (1.11) $y_0(s,\lambda)$ is the solution of the homogeneous system (1.9). This solution also satisfies conditions (1.6). Through $C(s,t)$ we denote Cauchy's kernel

$$C(s,t) = Y_1(s)Z_2(t) + Y_2(s)Z_4(t) \tag{1.12}$$

In the right side of (1.11) we take the integral containing w' by parts. We have (1.2), (1.3), (1.5).

$$\int_{s_0}^{s} C(s,t) A_0^{-1} A_4(t)\, d(t)\, dt = \int_{s_0}^{s} C(s,t) \begin{pmatrix} b_0 w' \\ 0 \end{pmatrix} dt = -Y_2(s)\, b_0(s_0) \begin{pmatrix} w(s_0) \\ 0 \end{pmatrix} - \int_{s_0}^{s} K(s,t) \begin{pmatrix} w(t) \\ 0 \end{pmatrix} dt \tag{1.13}$$

$$K(s,t) = [(Y_1(s) Z_2(t) + Y_2(s) Z_4(t))\, b_0(t)]_t' \tag{1.14}$$

In this connection the obvious relationships $C(s,s) = 0$, $C(s,s_0) = Y_2(s)$ were used. Thus we have

$$y(s,\lambda) = y_0(s,\lambda) - Y_2(s)\, b_0(s_0) \begin{pmatrix} w(s_0) \\ 0 \end{pmatrix} - \int_{s_0}^{s} K(s,t) \begin{pmatrix} w(t) \\ 0 \end{pmatrix} dt + \\ + \int_{s_0}^{s} C(s,t) A_0^{-1} A_5(t) \begin{pmatrix} w(t) \\ w(t) \end{pmatrix} dt \tag{1.15}$$

Differentiating this identity, we find

$$y'(s,\lambda) = y_0'(s,\lambda) - Y_2'(s)\, b_0(s_0) \begin{pmatrix} w(s_0) \\ 0 \end{pmatrix} - K(s,s) \begin{pmatrix} w(s) \\ 0 \end{pmatrix} + \int_{s_0}^{s} T(s,t) \begin{pmatrix} w(t) \\ w(t) \end{pmatrix} dt \tag{1.16}$$

Here $T(s,t)$ denotes a kernel which is continuous with respect to s and t, and whole with respect to λ. We now find $K(s,s)$. By virtue of the identity

$$Y_1(s) Z_2(s) + Y_2(s) Z_4(s) = 0 \tag{1.17}$$

we have for $K(s,s)$ from (1.14)

$$K(s,s) = [Y_1(s) Z_2'(s) + Y_2(s) Z_4'(s)]\, b_0(s) \tag{1.18}$$

Differentiating (1.17) and taking advantage of the fact that $Y_1'(s) Z_2(s) + Y_2'(s) Z_4(s) = E_2$ and using (1.8) and (1.10), we find easily

$$K(s,s) = -b_0(s) E_2 \tag{1.19}$$

From Eqs. (1.15), (1.16) it follows now that

$$u(s,\lambda) = u_0(s,\lambda) - u_1(s,\lambda)\, b_0(s_0)\, w(s_0) + \int_{s_0}^{s} K_1(s,t)\, w(t)\, dt$$

$$v(s,\lambda) = v_0(s,\lambda) - v_1(s,\lambda)\, b_0(s_0)\, w(s_0) + \int_{s_0}^{s} K_2(s,t)\, w(t)\, dt \tag{1.20}$$

$$u'(s, \lambda) = u_0'(s, \lambda) - u_1'(s, \lambda) b_0(s_0) w(s_0) + b_0(s) w(s) + \int_{s_0}^{s} K_3(s, t) w(t)\, dt$$

Here (u_1, v_1) are solutions of the one-dimensional system (1.7) such that

$$u_1(s_0) = v_1(s_0) = v_1'(s_0) = 0, \qquad u_1'(s_0) = 1 \tag{1.21}$$

while K_1, K_2 and K_3 are kernels which are continuous with respect to s and t and complete with respect to λ. Substituting u, v u' from (1.20) into (1.4), we obtain the following equation with respect to $w(s, \lambda)$:

$$(\varphi_2(s) - \lambda) w(s, \lambda) + \int_{s_0}^{s} K_4(s, t) w(t, \lambda)\, dt = r(s, \lambda) \tag{1.22}$$

where

$$r(s, \lambda) = b_0(s) u_0'(s) + b_1(s) u_0(s) + b_2(s) v_0(s) -$$
$$- [b_0(s) u_1'(s) + b_1(s) u_1(s) + b_2(s) v_1(s)] b_0(s_0) w(s_0) \tag{1.23}$$

and

$$w(s_0) = \frac{b_0(s_0) r_3 + b_1(s_0) r_1 + b_0(s_0) r_2}{\varphi_2(s_0) - \lambda} \tag{1.24}$$

Equation (1.24) is obtained by substituting $s = s_0$ in (1.22). Under the assumptions made with respect to λ, Eq. (1.22) always has a unique solution which is analytic in λ. Therefore, substituting the quantity $w(s, \lambda)$ found in this manner, into (1.20), we determine the solution for the Cauchy problem. The latter is apparently unique. The proof of the lemma is complete.

Note 1.1. If r_1, r_2, r_3 are such that

$$b_0(s_0) r_3 + b_1(s_0) r_1 + b_2(s_0) r_2 = 0 \tag{1.25}$$

the solution of the Cauchy problem is regular also for $\lambda = \varphi_2(s_0)$.

Note 1.2. If the values r_1, r_2, r_3 are such that (1.25) is satisfied, the Cauchy problem for $\lambda = \varphi_2(s_0)$ and conditions (1.6) has a unique solution, if additionally an arbitrary value is given for $w(s_0)$.

2. Nature of the spectrum in the momentless case. Lemma 2.1. The spectrum of the boundary value problem (0.8), (0.7) is real and nonnegative. The entire interval $[\alpha, \beta]$ consists of points of the spectrum. Outside the interval $[\alpha, \beta]$ the spectrum is discrete (it consists of isolated eigenvalues of finite multiplicity). The limit points of the discrete spectrum is $\lambda = +\infty$ and, perhaps, the end points of the interval $[\alpha, \beta]$.

Proof. Let us examine the inhomogeneous equation

$$L_0 f - \lambda f = h \tag{2.1}$$

Here $h = (h_1(s), h_2(s), h_3(s))$ is an arbitrary vector function, which is integrable in the square; $f = (u, v, w)$ is the unknown vector function which satisfies the conditions (0.7). The system (2.1) can be written in the following form:

$$l_\lambda y = p + h^* \tag{2.2}$$

$$-b_0(s) u' - b_1(s) u - b_2(s) v + (\varphi_2(s) - \lambda) w = h_3$$

where the operator l_λ and vector p are the same as in (1.7) and $h^* = (h_1, h_2)$. Let $G(s, t, \lambda)$ be the Green function of operator l_λ for boundary conditions (0.7), then

$$y = \int_a^b G(s, t, \lambda)\,[p(t) + h^*(t)]\,dt \tag{2.3}$$

We note that

$$G(s, t, \lambda) = \begin{cases} Y_1(s)\,Z_2(t) & \text{for} \quad t \leqslant s \\ -Y_2(s)\,Z_4(t) & \text{for} \quad t \geqslant s \end{cases} \tag{2.4}$$

We denote by $Y_1(s)$ and $Y_2(s)$ 2 x 2 matrices which satisfy the equation $l_\lambda(y) = 0$ and the following conditions:

$$Y_1(b, \lambda) = 0, \quad Y_1'(b, \lambda) = E_2, \quad Y_2(a, \lambda) = 0, \quad Y_2'(a, \lambda) = E_2 \tag{2.5}$$

By Z_2 and Z_4 we denote blocks of matrix $Z(t, \lambda)$, which is inverse to

$$Y(t, \lambda) = \begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix}$$

The matrix $Z(t, \lambda)$ is meromorphic, its poles are points of the spectrum of the operator l_λ. They are positive and tend to $+\infty$.

Integrating in (2.3) the term containing w' by parts and substituting the expression for u and v obtained in this manner into the second equation (2.2), we obtain in analogy to (1.22)

$$(\varphi_2(s) - \lambda)\,w(s) + \int_a^b Q_1(s, t, \lambda)\,w(t)\,dt = \int_a^b Q_2(s, t, \lambda)\,h_1(t)\,dt + \int_a^b Q_3(s, t, \lambda)\,h_2(t)\,dt + h_3(s). \tag{2.6}$$

Here

$$Q_i(s, t, \lambda) \qquad (i = 1, 2, 3) \tag{2.7}$$

are functions which are continuous with respect to s and t and meromorphic with respect to λ with poles at the points of the spectrum of the operator l_λ.

In this manner Eq. (2.6) is boundedly solvable (*) outside of these poles for any right side h if and only if Eq. (2.1) is boundedly solvable and consequently the spectra of Eqs. (2.1) and (2.6) coincide outside the poles $G(s, t, \lambda)$. Utilizing theorems of functional analysis [10], it is easy to show that the spectrum of Eq. (2.6) consists of the interval $[\alpha, \beta]$ and a desorete set.

The reality of the spectrum follows from the self-adjoint property of the operator L_0, The nonnegativity follows from the condition $L_0 \geqslant 0$. The fact that $\lambda = +\infty$ is always the limit point of the spectrum of the operator L_0, and can easily be established by the examination of the asymptotics of solutions of system (0.8) when $\lambda \to +\infty$. Lemma 2.1 is proven.

Let us examine more closely the discrete spectrum of operator L_0. Let

$$f^{(1)}(s, \lambda), \qquad f^{(2)}(s, \lambda) \tag{2.8}$$

(*) The inhomogeneous equation (2.1) is boundedly solvable for a given λ, if for any smooth right side $h(s)$ it has a smooth solution $f(s)$, satisfying the boundary condition (0.7) and also $(f, f) \leqslant C_0\,(h, h)$, where the constant C_0 is independent of h. An analogous statement applies to Eq. (2.6). The spectrum coincides with the set of those values of λ, for which the bounded solvability is violated.

be two solutions of Eq. (0.8) which satisfy Cauchy's initial conditions

$$u_1(a,\lambda)=0,\quad v_1(a,\lambda)=0,\quad u_1'(a,\lambda)=1,\quad v_1'(a,\lambda)=0$$
$$u_2(a,\lambda)=0,\quad v_2(a,\lambda)=0,\quad u_2'(a,\lambda)=0,\quad v_2'(a,\lambda)=1 \tag{2.9}$$

for all

$$\lambda \in [\alpha,\ \beta],\quad \lambda \neq \varphi_2(a) \tag{2.10}$$

Taking into consideration Lemma 1.1, it is easy to show that the discrete spectrum of the operator L_0 on the set (2.10) coincides with the zeros of the function

$$\Delta(b,\lambda)=\mathrm{Det}\begin{pmatrix} u_1(b,\lambda) & u_2(b,\lambda) \\ v_1(b,\lambda) & v_2(b,\lambda)\end{pmatrix} \tag{2.11}$$

which is analytic on the set (2.10) and has a pole perhaps only at the point $\lambda=\varphi_2(a)$

We note in conclusion that all eigenvalues which satisfy condition (2.10) can be no more than twofold eigenvalues because by virtue of Lemma 1.1 each eigenfunction $f(s,\ \lambda_k)$ is a linear combination of solutions (2.8) with $\lambda=\lambda_k$. The point $\lambda=\varphi_2(a)$ can be no more than a threefold eigenvalue (Note 1.2 to Lemma 1.1).

3. Oscillation theorem. Let us examine the zeros of the function

$$\Delta(s,\lambda)=\mathrm{Det}\begin{pmatrix} u_1(s,\lambda) & u_2(s,\lambda) \\ v_1(s,\lambda) & v_2(s,\lambda)\end{pmatrix}\qquad (a\leqslant s\leqslant b) \tag{3.1}$$

where $u_i(s,\ \lambda)$ and $v_i(s,\ \lambda)$ are components of solutions (2.8).

Lemma 3.1. For fixed $a<s_0\leqslant b$ the zeros of the function $\Delta(s_0,\ \lambda)$ in the region

$$\lambda \in [\alpha,\ \beta],\qquad \lambda \neq \varphi_2(a) \tag{3.2}$$

are no more than double valued. In other words if $\Delta(s_0,\ \lambda_0)=0$ and $(\partial/\partial\lambda)\times$ $\times\ \Delta(s_0,\ \lambda_0)=0$, then $(\partial^2/\partial\lambda^2)\ \Delta(s_0,\lambda_0)\neq 0$. In this connection the multiplicity of λ_0 as a root of equation $\Delta(s_0,\ \lambda)$ coincides with the multiplicity of the eigenvalue of the differential equation

$$L_0 f=\lambda f\qquad (a\leqslant s\leqslant s_0) \tag{3.3}$$

with the boundary conditions

$$u(a)=u(s_0)=v(a)=v(s_0)=0 \tag{3.4}$$

We present only the idea of the proof. The fact that zeros of the function $\Delta(s_0,\ \lambda)$ are eigenvalues of the problem (3.3), (3.4) is obvious. Through infinitely small perturbation of the lowest terms (even those not containing derivatives) of the equation $L_0 f=\lambda f$ it is possible to make all zeros of $\Delta(s_0,\ \lambda)$ simple in the vicinity of the point λ_0. According to Rouche's theorem the number of these zeros is equal to the multiplicity of the root λ_0 of the equation $\Delta(s_0,\ \lambda)=0$. In this case it is easy to select the perturbation so that all eigenvalues of the boundary value problem also become simple. According to well-known perturbation theory the number of eigenvalues arising in this manner is equal to the multiplicity of λ_0, as eigenvalue. Lemma 3.1 follows from this. The following statement is also valid.

Lemma 3.2. Let λ_0 be a single root of the equation

$$\Delta(s_0,\ \lambda)=0 \tag{3.5}$$

in the region (3.2). Then in the vicinity of the point $s=s_0$ the equation $\Delta(s,\ \lambda)=$ $=0$ determines one differentiable function $\lambda=\lambda(s)(\lambda(s_0)=\lambda_0)$. In this connection

$$\left.\frac{d\lambda}{ds}\right|_{s=s_0} = - B(s_0)\left\{\frac{\varphi_1(s_0)-\lambda_0}{\varphi_2(s_0)-\lambda_0} u_0'^2(s_0) + \frac{1-\sigma}{2} v_0'^2(s_0)\right\} \tag{3.6}$$

Here $f_0(s) = (u_0(s), v_0(s), w_0(s))$ is the eigenvector of the problem (3.3),(3.4) normalized by the condition

$$\int_a^b B(s)(u_0^2 + v_0^2 + w_0^2)\,ds = 1 \tag{3.7}$$

For $\lambda_0 = \varphi_2(s_0)$ the first term in braces (3.6) should be replaced by zero.

Note 3.1. If λ_0 is a double root of Eq.(3.5), then equation $\Delta(s, \lambda) = 0$ determines two differentiable functions $\lambda_1(s)$ and $\lambda_2(s)$ such that $\lambda_1(s_0) = \lambda_2(s_0) = \lambda_0$. Equation (3.6) is valid for derivatives $d\lambda_1/ds$ and $d\lambda_2/ds$. In the right side of this equation two eigenvectors of problem (3.3),(3.4) appear. These eigenvectors are orthogonal to each other and correspond to $\lambda = \lambda_0$.

Proof. Together with problem (3.3),(3.4) let us examine the "perturbed" problem

$$L_0(z) f(z) = \lambda f(z) \qquad (a \leqslant z \leqslant s_0 + \varepsilon) \tag{3.8}$$

Here the conditions at the ends of the interval $[a, s_0 + \varepsilon]$ are assumed to be the same as in (3.4), ε is the small parameter. We take advantage of the fact that roots of the equation $\Delta(s_0 + \varepsilon, \lambda) = 0$ represent eigenvalues of problem (3.8). Derivatives of eigenvalues of this problem with respect to parameter ε can be found by means of equations of the perturbation theory of linear operators. We make the following substitution into (3.8)

$$z = s + \varepsilon \frac{s-a}{s_0 - a} \qquad (a \leqslant s \leqslant s_0) \tag{3.9}$$

As a result the operator $L_0(z)$ in (3.8) can be represented in the form

$$L_0(z) = L_0(s) + \varepsilon L^{(1)}(s) + O(\varepsilon^2) \quad (a \leqslant s \leqslant s_0) \tag{3.10}$$

According to known theorems of perturbation theory [7] eigenvalues of the boundary value problem (3.8) are differentiable functions of ε, and the following equation is valid (*)

$$\left.\frac{d\lambda}{d\varepsilon}\right|_{\varepsilon=0} = (L^{(1)} f_0, f_0) \tag{3.11}$$

where $f_0(s)$ is the normalized eigenvector of the operator $L_0(s)$ $(a \leqslant s \leqslant s_0)$. Equation (3.11) is valid also in the case when λ_0 is a double valued point of the spectrum. It is valid for both functions $\lambda_k(\varepsilon)$ $(k = 1, 2)$. For calculation of the right side of (3.11) it is useful to make the substitution (3.9) in the following identity:

$$L_0(z) f_0(z) = \lambda_0 f_0(z) \quad (a \leqslant z \leqslant s_0) \tag{3.12}$$

In the relationship obtained in this manner, we separate terms of the first order in ε, and find that

(*) By $O(\varepsilon^2)$ in Eq.(3.10) we understand a differential operator which contains the factor ε^2. It is possible to show that operators $L_0^{-1}(s) L^{(1)}(s)$ and $L_0^{-1}(s) O(\varepsilon^2)$ are bounded. This makes it possible to apply the theorem on perturbed eigenvalues from the cited book [7].

$$L^{(1)}(s)\, f_0(s) + L_0(s)\left(f_0'(s)\frac{s-a}{s_0-a}\right) = \lambda_0 f_0'(s)\frac{s-a}{s_0-a} \tag{3.13}$$

From here according to (3.11)

$$\frac{d\lambda}{d\varepsilon}\Big|_{\varepsilon=0} = -(L_0(s)\, g, f_0) + \lambda_0(g, f_0) \qquad g(s) = f_0'(s)\frac{s-a}{s_0-a} \tag{3.14}$$

We note that

$$\lambda_0(g, f_0) = (g, L_0(s) f_0)$$

Since $g(s)$ does not satisfy boundary conditions (3.4) only on the right end of the interval $[a, s_0]$ then in the right side of (3.14) on integration by parts of the first term, only the terms outside the integral are preserved for $s = s_0$. Turning to the explicit expressions in the left side of (0.1), it is easy to find that those contributions are different from zero which are obtained through integration by parts of the following expressions:

$$-\int_a^{s_0} B(s)\frac{d}{ds}\left(\frac{1}{B}\frac{d(Bg_1)}{ds}\right)u_0(s)\, ds$$

$$-\int_a^{s_0} \frac{1-\sigma}{2} B\frac{d}{ds}\left(\frac{1}{B}\frac{d}{ds}(Bg_2)\right)v_0(s)\, ds$$

$$-\int_a^{s_0} B(s)(R_1^{-1}(s) + \sigma R_2^{-1})\frac{dg_1}{ds} w_0(s)\, ds$$

Adding these expressions, we obtain

$$\frac{d\lambda}{d\varepsilon}\Big|_{\varepsilon=0} = -B(s_0)\left(u_0''(s_0) + \frac{1-\sigma}{2}v_0''(s_0) - u_0'(s_0)\, b_0(s_0)\, w_0(s_0)\right)$$

Here we substitute $w_0(s)$ from Eq. (1.4). For the condition $\lambda_0 \neq \varphi_2(s_0)$ we obtain Eq. (3.6). However, if $\lambda_0 = \varphi_2(s_0)$, then $u_0'(s_0) = 0$ and only the second term is preserved in braces (3.6). Lemma 3.2 is proven.

Lemma 3.3. Let $\lambda \in [\alpha, \delta]$. Then all roots of Eq. (3.16) in the half-interval $(a, b]$ of variation of s with decreasing λ are displaced to the right.

Proof. We note that if $\lambda \in [\alpha, \delta]$ then in Eq. (3.6) both terms are nonnegative. By virtue of Lemma 1.1 at least one of the terms is different from zero. Therefore for the condition $\lambda \in [\alpha, \delta]$

$$d\lambda/ds < 0 \tag{3.15}$$

Consequently, in some sufficiently small two-dimensional neighborhood of the point (s_0, λ_0) all points (s, λ), which satisfy the equation

$$\Delta(s, \lambda) = 0 \tag{3.16}$$

form maps of no more than two smooth monotonically decreasing functions of $\lambda(s)$. The inverse quantities $s(\lambda)$ of these functions are continuous and decreasing. This leads to the proof of Lemma 3.3.

We note in passing that for fixed $\lambda < 0$ Eq. (3.16) does not have any roots at all in the half-interval $s \in (a, b]$. This follows from the positive definiteness of the operator (3.3) for any $a < s_0 \leqslant b$. In conclusion we shall amplify our considerations about zeros of Eq. (3.16) through the following remark.

Lemma 3.4. All zeros of Eq. (3.16) for fixed $\lambda \in [\alpha, \delta]$ are no more than double valued. If

$$\Delta(s_0, \lambda_0) = 0, \qquad \partial\Delta/\partial s(s_0, \lambda_0) = 0 \tag{3.17}$$

then

$$\partial^2/\partial s^2\, \Delta(s_0, \lambda_0) \neq 0 \tag{3.18}$$

In this connection a double valued point of the spectrum of problem (3.3), (3.4) corresponds to a double root s_0.

Proof. If (3.17) is satisfied, then it follows from the obvious equality $\Delta_s' + \Delta_\lambda' \cdot \lambda_s' = 0$ that $\Delta_\lambda' = 0$ if we take into account that $\lambda_s' \neq 0$. Since $\Delta_{\lambda\lambda} \neq 0$(Lemma 3.1), the negative values of both derivatives in (3.6) coincide with roots of the quadratic equation $\Delta_{ss}'' + 2\Delta_{s\lambda}''\xi + \Delta_{\lambda\lambda}''\xi^2 = 0$. Equation (3.18) follows from this.

Let us renumber the eigenvalues of problem (0.8), (0.7), which are less than α [6], in increasing order taking into account the multiplicity

$$\lambda_0^{(1)} \leqslant \lambda_1^{(1)} \leqslant \ldots \leqslant \lambda_k^{(1)} \leqslant \ldots \tag{3.19}$$

Let $n_1(\lambda)$ be the number of zeros of function $\Delta(s, \lambda)$ for fixed λ in the half-interval $(a, b]$. The following oscillation theorem is the result of analyses performed above.

Theorem 3.1. a) The number of eigenvalues of problem (0.8), (0.7) which do not exceed λ, is equal to $n_1(\lambda)$ $(\lambda < \alpha)$.

b) The first series is infinite if an only if

$$\sup_{\lambda<\alpha} n_1(\lambda) = \infty$$

c) $\lambda_k^{(1)}$ is a multiple eigenvalue if and only if $s = b$ is a double zero of function $\Delta(s, \lambda_k^{(1)}$)d) If $\lambda_k^{(1)}$ is a simple eigenvalue, then in the interval (a, b) function $\Delta(s, \lambda_k^{(1)})$ has exactly k zeros taking into account the multiplicities.

4. Conditions for infinity of the first series. In this section we will point out the conditions placed on $B(s)$, for which the problem (0.8), (0.7) leads to an infinite series of eigenvalues less than α, where

$$\alpha = \inf \varphi_1(s) \qquad (s \in [a, b]) \tag{4.1}$$

The corresponding theorems are generalizations of conditions given in the axisymmetric case [6, 9].

Expressing $w(s)$ in the third equation of system (0.1) through u, v and u and substituting it into the first two, we arrive at a system of two equations

$$\begin{gathered} C_1u'' + C_2u' + C_3v' + C_4u + C_5v = 0 \\ D_1v'' + D_2u' + D_3v' + D_4u + D_5v = 0 \end{gathered} \tag{4.2}$$

Here C_i and D_i are functions of s and λ, The explicit expressions for these functions have not been presented so far. It follows from the oscillation theorem that the first series can be infinite only if for $\lambda = \alpha$ the system (4.2) has a solution with an infinite number of zeros. This situation does not contradict the uniqueness theorem because for $\lambda = \alpha$ the system (4.2) has a singularity. It is actually easy to check that

$$C_1(s; \alpha) = B(s)\frac{\varphi_1(s) - \alpha}{\varphi_2(s) - \alpha}$$

Therefore the points s_0, for which

$$\varphi_1(s_0) = \alpha \tag{4.3}$$

are singular.

Let us examine the asymptotics of solutions of system (4.2) in the vicinity of the singular point s_0. We note that all functions C_i and D_i $(i = 1, 2, \ldots, 5)$ are smooth functions of s and that $D_1(s, \alpha) \neq 0$. If s_0 is the end of the interval $[a, b]$ and $\varphi_1'(s_0) \neq 0$, then it is easy to show that solutions of system (4.2) have only a finite number of zeros. Infinite oscillation is possible only for the condition

$$\varphi_1'(s_0) = 0 \tag{4.4}$$

which by virtue of (4.1) always occurs when s_0 is an inside point of the interval $[a, b]$. We recall that $\varphi_1(s) = (1-\sigma)^2 B^{-2}(1 - B'^2)$. Let

$$B(s) = \beta_0 + \beta_1 (s - s_0) + {}^1/_2\, \beta_2 (s - s_0)^2 + {}^1/_6\, \beta_3 (s - s_0)^3 + \ldots \tag{4.5}$$

Here we have introduced obvious notation for derivatives of function $B(s)$ at the point s_0. We note that $|\beta_1| = |B'(s_0)| \leqslant 1$. For $\varphi_1(s)$ we have the expansion

$$\varphi_1(s) = \frac{1-\sigma^2}{\beta_0^2}\{\omega_0 + \omega_1(s - s_0) + {}^1/_2\, \omega_2 (s - s_0)^2 + \ldots\} \tag{4.6}$$

where

$$\omega_0 = 1 - \beta_1^2, \qquad \omega_1 = -2\beta_1\left(\beta_2 + \frac{1-\beta_1^2}{\beta_0}\right) \tag{4.7}$$

$$\omega_2 = 2\left[-\beta_2^2 - \frac{\beta_2}{\beta_0}(1 - 5\beta_1^2) + \frac{3(1-\beta_1^2)\beta_1^2}{\beta_0^2} - \beta_1\beta_3\right]$$

In this manner the condition (4.4) is satisfied in two cases

$$\text{a)}\ \ \beta_1 = 0 \tag{4.8}$$

$$\text{b)}\ \ \beta_2 + (1 - \beta_1^2)/\beta_0 = 0 \tag{4.9}$$

which we will examine separately. At present let us assume that

$$\omega_2 \neq 0 \tag{4.10}$$

Coefficients of system (4.2) are expanded in powers of $s - s_0 = t$ in the vicinity of the point s_0. As a result we obtain

$$\begin{aligned}(a_1 t^2 + o(t^2))\, u'' + (2a_1 t + o(t))\, u' + (a_3 + o(1))\, u +\\ + (b_1 + o(1))\, v' + (b_2 + o(1))\, v = 0\\ (-b_1 + o(1))\, u' + (c_2 + o(1))\, u + d_1 v'' + (d_2 + o(1))\, v' +\\ + (d_3 + o(1))\, v = 0, \quad t = s - s_0\end{aligned} \tag{4.11}$$

Expressions for coefficients of Taylor expansions are presented below. Substituting $u = y_1$, $v = y_2$, $u_t' = 1/t\,(y_3)$, and $v_t' = y_4$ the system (4.11) is reduced to a system of four first order equations. The system is written directly in the matrix-vector form

$$y' = \left(\frac{1}{t}\Omega_0 + \Omega_1 + o(1)\right) y \tag{4.12}$$

Here $y = (y_1, y_2, y_3, y_4)$ is the unknown column vector Ω_0, Ω_1 are constant matrices. In this case

$$\Omega_0 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ -a_3/a_1 & -b_2/a_1 & -1 & -b_1/a_1\\ 0 & 0 & 2b_1/(1-\sigma) & 0 \end{pmatrix} \tag{4.13}$$

It is easy to check that

$$\operatorname{Det}(\Omega_0 - \mu E_4) = \mu^2\left\{\mu^2 + \mu + \frac{2b_1^2}{a_1(1-\sigma)} + \frac{a_3}{a_1}\right\} \tag{4.14}$$

Let $\mu_1,\ \mu_2,\ \mu_3 = \mu_4 = 0$ be the roots of the characteristic polynomial (4.14). The corresponding eigenvectors of the matrix Ω_0 have the form

$$f^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \mu_1 \\ \frac{2b_1}{1-\sigma} \end{pmatrix},\quad f^{(2)} = \begin{pmatrix} 1 \\ 0 \\ \mu_2 \\ \frac{2b_1}{1-\sigma} \end{pmatrix},\quad f^{(3)} = \begin{pmatrix} b_1 \\ 0 \\ 0 \\ -a_3 \end{pmatrix},\quad f^{(4)} = \begin{pmatrix} 0 \\ b_1 \\ 0 \\ -b_2 \end{pmatrix} \tag{4.15}$$

It is easy to show that the system (4.12) has four linear independent solutions of the form

$$y^{(1)}(t) = t^{\mu_1} f^{(1)} (1 + o(1)),\quad y^{(2)}(t) = t^{\mu_2} f^{(2)} (1 + o(1)),\quad y^{(3)}(t) = f^{(3)} + o(1)$$
$$y^{(4)}(t) = f^{(4)} + o(1) \tag{4.16}$$

Infinite oscillation in the vicinity of the point $s = s_0$ is possible only for the condition where the roots μ_1 and μ_2 of the polynomial (4.14) are nonreal. The corresponding condition has the form

$$D = a_1^2 - 4a_1 (b_1^2/d_1 + a_3) < 0 \tag{4.17}$$

Now we shall prove the following proposition.

Theorem 4.1. Let at some point $s = s_0$ $\varphi_1(s_0) = \alpha > 0$ and the condition (a) $B'(s_0) = 0$ be satisfied (4.8). We set

$$B(s) = B_0 - \frac{k}{2}(s - s_0)^2 + o((s - s_0)^3)$$

and let

$$0 < kB_0 < 1 \tag{4.18}$$

Then for the condition

$$9(kB_0)^2 + (12\sigma - 1)kB_0 + 4\sigma^2 + 8m^2(1 - kB_0) > 0 \tag{4.19}$$

α is the limit point of the discrete spectrum of the problem (0.8), (0.7) (the first series is infinite).

Note 4.1. Here $k = |B''(s_0)|$ is the curvature of the meridian at the point $s = s_0$. The inequality (4.18) indicates that for $s = s_0$ the abscissa of the meridian reaches a maximum and the center of curvature lies on the other side of the axis of rotation (Fig. 1).

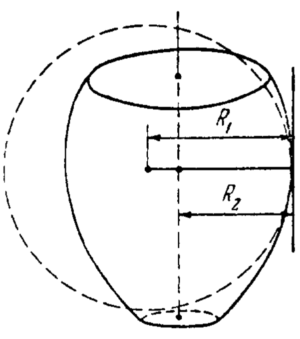

Fig. 1.

Note 4.2. For $\sigma > 1/24$ the condition (4.19) is superfluous.

Proof. Relationship (4.10) is valid for conditions (4.8) and (4.18) (see (4.7)). For the coefficients of Taylor expansions in (4.11) we have

$$a_1 = \frac{(1-\sigma^2)k(1-kB_0)}{B_0(B_0k+\sigma)^2} \neq 0$$

$$a_3 = -\frac{m^2(1-\sigma)^2}{2B_0^2} + (1-\sigma^2)\frac{2B_0k+\sigma}{B_0^2(B_0k+\sigma)} \tag{4.20}$$

$$b_1 = \frac{m(1-\sigma)}{2B_0(B_0k+\sigma)}(B_0k - \sigma - 2) \neq 0,\qquad d_1 = \frac{1-\sigma}{2}$$

In this case b_2, c_2, d_2 and d_3 are some constants which are unessential to the problem under consideration. It is easy to verify that the condition of oscillation (4.17) takes the form

$$-\frac{(1-\sigma^2)^2(1-kB_0)k}{B_0^3(B_0k+\sigma)^4}\{9(kB_0)^2+(12\sigma-1)kB_0+4\sigma^2+8m^2(1-kB_0)\}<0$$

As a consequence of (4.18) this is equivalent to the requirement (4.19). Therefore Eq. (4.14) has two nonreal roots

$$\mu_1=\bar{\mu}_2=-{}^1/_2+i\beta, \qquad \beta\neq 0 \tag{4.21}$$

Let us assume for simplicity of presentation that $s_0\neq a$ and that in the half-interval $[a, s_0)$ the system (4.2) does not have any singular points (*) $C_1(s, \alpha)\neq 0$ for $a\leqslant s<s_0$. Let the matrix

$$Y(s)=\begin{pmatrix}Y_1(s) & Y_2(s)\\ Y_3(s) & Y_4(s)\end{pmatrix}, \qquad s\in[a, s_0) \tag{4.22}$$

be composed of solutions (4.16). It is evident that $Y(a)$ is nondegenerate. We write

$$Y^{-1}(a)=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

Through $\Delta(s)$ we denote the upper right minor of the matrix $Y(s)Y^{-1}(a)$. It is easy to verify that for $s\to s_0-0$

$$\Delta(s)=\operatorname{Re}[(s-s_0)^{\mu_1}(b_{11}d_{22}-b_{12}d_{21})b_1]+o(1) \tag{4.23}$$

Here b_{ij} and d_{ij} are elements of matrices B and D, and $b_1\neq 0$ as in (4.20). If now

$$b_{11}d_{22}-b_{12}d_{21}\neq 0 \tag{4.24}$$

then in the left half-neighborhood of point s_0 the function $\Delta(s)$ has infinite number of zeros.

Let N be an arbitrary integer. Let us select $\delta>0$ so small that the function $\Delta(s)$ will have N zeros in the half-interval $[a, s_0-\delta)$. Then we select a $\lambda<\alpha$ so close to α, that the function $\Delta(s, \lambda)$ which was introduced by Eq. (3.1), for the system (4.2) will have N zeros in the half-interval $[a, s_0-\delta)$. This can be done on the basis of the theorem on continuous dependence of the solution of the Cauchy problem on the parameter λ. Since N is arbitrary, the first series is infinite according to Theorem 3.1 and the proof for Theorem 4.1 is complete.

It remains to be noted that condition (4.24) can be discarded. It is actually possible to show that an arbitrarily small perturbation of system (0.1) always exists such that in the perturbed system the condition (4.24) will be satisfied. This condition is realized with the aid of the positive definite matrix which is equal to zero outside the right ε-half-neighborhood of the point a. Consequently, the perturbed system will have an infinite first series. Since the indicated perturbation increases the eigenvalues, it is established by the same token that the unperturbed initial problem also has an infinite first series.

The following proposition is also valid.

Theorem 4.2. Let at some point $s=s_0$ $\varphi_1(s_0)=\alpha>0$ and let condition (b) be satisfied (4.9)

$$\beta_2\beta_0+1-\beta_1^2=0 \tag{4.25}$$

where β_s are coefficients of the Taylor expansion for the function $B(s)$ in the vicinity $s=s_0$ as in (4.5.).

(*) The reader can easily extend tne proof presented below to the general case.

Let also (*)

$$\omega_2 = \frac{\beta_2\beta_1^2 - \beta_0\beta_1\beta_3}{\beta_0} \neq 0 \tag{4.26}$$

Then α is the limit point of the discrete spectrum of problem (0.7), (0.8).

Proof. It was already mentioned that for condition (4.25) $\varphi_1'(s_0) = 0$ (see (4.7)). In this connection it can be verified that Taylor's coefficients in system (4.11) have the following form:

$$a_1 = \frac{1-\sigma}{2(1+\sigma)}\frac{\omega_2}{\omega_0}$$

$$a_3 = (1-\sigma)\left[(2+\sigma)\frac{\omega_0}{\beta_0^2} + \frac{\omega_2}{2(1+\sigma)\,\omega_0}\right] - \frac{(1-\sigma)\,m^2}{2\beta_0^2} \tag{4.27}$$

$$b_1 = -\frac{m\,(1-\sigma)}{2\beta_0}, \qquad {}_1 = \frac{1-\sigma}{2}$$

Substitution of these expressions into (4.17) leads to the inequality

$$\frac{1-\sigma}{1+\sigma}\frac{\omega_2}{\omega_0}\left[-4\,(1-\sigma)(2+\sigma)\,\frac{\omega_0}{\beta_0^2} - \frac{\omega_2}{2\omega_0}\frac{3(1-\sigma)}{1+\sigma}\right] < 0$$

which is always satisfied by virtue of inequality $\omega_2 > 0$. The proof is carried out in the same manner as in Theorem 4.1.

Let us explain that for conditions (4.25) and $\omega_2 > 0$ the plot of the meridian in the vicinity of the point $s = s_0$ has the form represented in Fig. 2 (we assume that $0 < B'(s_0) < 1$). The dashed line indicates the adjoining circle. For condition (4.25) $R_1 = R_2$, i.e. the corresponding point is umbilical. In conclusion we note the following fact.

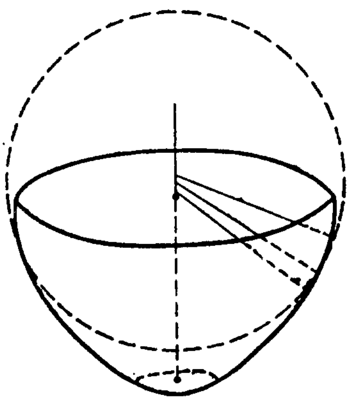

Fig. 2.

Theorem 4.3. Let for some $s = s_0$ function $\varphi_1(s_0) = \alpha > 0$, let $p > 1$ and

$$\varphi_1(s) = \varphi_1(s_0) + \frac{\varphi_1^{(2p)}(s_0)}{(2p)!}(s-s_0)^{2p} + o\,((s-s_0)^{2p}) \tag{4.28}$$

where

$$\varphi_1^{(2p)}(s_0) > 0 \tag{4.29}$$

Then α is the limit point of eigenvalues. The proof of this fact must be omitted due to lack of space.

Appendix. Proof of Eq. (0.10). Let us denote through $\mathrm{L}_2(a, b)$ the Hilbert space of vector function $g(s) = (g_1, g_2, g_3)$ with scalar product (0.5) and let $L_2(a, b)$ be a space of scalar functions with an integrable square. For fixed $g \in \mathrm{L}_2(a, b)$ we introduce the function

$$\varphi(\mu) = (L_\mu^{-1} g, g) \tag{A.1}$$

Let $\mu_0 > 0$, then we have

$$\frac{\Delta\varphi}{\Delta\mu} = \frac{1}{\Delta\mu}\left((L^{-1}_{\mu_0+\Delta\mu} - L^{-1}_{\mu_0})\,g, g\right) = -\frac{1}{\Delta\mu}\left(L^{-1}_{\mu_0+\Delta\mu}(L_{\mu_0} - L_{\mu_0+\Delta\mu})\,L^{-1}_{\mu_0} g, g\right) = \tag{A.2}$$

(*) Since $\alpha = \operatorname{Inf}\varphi_1(s)$, condition (4.26) implies the inequality $\omega_2 > 0$.

$$= -\frac{\Delta\mu^4}{\Delta\mu}(L^{-1}_{\mu_0+\Delta\mu}L_4L^{-1}_{\mu_0}g, g)$$

by L_4 we denote here the operator which in the left side of system (0.1) stands with μ^4. The operator L_4 contains fourth order differentiation of the coordinate $w(s)$. It is self-adjoint and nonnegative for conditions (0.3). We note also that the operator $L_4L^{-1}_{\mu_0}$ is bounded. Therefore, going to the limit in (A.2) for $\Delta\mu \to 0$, we obtain

$$d\varphi/d\mu = -4\mu_0^3(L^{-1}_{\mu_0}L_4L^{-1}_{\mu_0}g, g) \leqslant 0 \qquad (A.3)$$

Consequently $\varphi(\mu)$ decreases monotonically and according to [7] the vector function $L^{-1}_{\mu}g$ converges in the sense of the norm $L_2(a, b)$ for $\mu \to 0$. Now we shall prove (0.10) It is sufficient to show that

$$\lim_{\mu\to 0}((L^{-1}_{\mu} - L^{-1}_0)g, h) = 0 \qquad (A.4)$$

for an everywhere dense set of vectors h in $L_2(a, b)$. Noting that

$$(L^{-1}_{\mu} - L^{-1}_0)g = -\mu^4 L^{-1}_0 L_4 L^{-1}_{\mu}g \qquad (A.5)$$

we take h in (A.4) in such a manner that the vector function $L_0^{-1}h$ will be sufficiently smooth and will satisfy the boundary conditions (0.3). Then

$$((L^{-1}_{\mu} - L^{-1}_0)g, h) = -\mu^4(L^{-1}_{\mu}g, L_4L^{-1}_0h)$$

and consequently (A.4) is valid.

It remains to establish that the set of such vector functions h is everywhere dense in $L_2(a, b)$. For this we note that for $h = 0$ the integral equation (2.6) for $\lambda = 0$ has only the trivial solution $w(s) \equiv 0$. Otherwise $\lambda = 0$ would be an eigenvalue of the problem (0.7),(0.8), which is not valid according to the initial assumption. Therefore the integral operator which is in the left side of (2.6) transforms the everywhere dense in $L_2(a, b)$ set of scalar functions into an everywhere dense set. The set of finite (infinitely differentiable) scalar functions $w(s)$ is everywhere dense in $L_2(a, b)$. Sets of corresponding functions in the right sides of (2.6) have this property. From this it is now obvious that given arbitrary continuous $h_1(s)$ and $h_2(s)$, it is always possible to find a function $h_3(s)$ from the everywhere dense set in $L_2(a, b)$, so that the vector function L^{-1}_0h will be sufficiently smooth and will satisfy conditions (0.3).

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BIBLIOGRAPHY

1. Gol'denveizer, A. L. Some mathematical problems of the linear theory of thin shells. Usp. Matem. Nauk Vol. 15, No. 5, 1960.
2. Gol'denveizer, A. L. Asymptotic integration of linear partial differential equations with a small principal part. PMM Vol. 23, No. 1, 1959.
3. Gol'denveizer, A. L. Qualitative analysis of free vibrations of an elastic thin shell. PMM Vol. 30, No. 1, 1966.
4. Alumäe, N. A. On the fundamental system of integrals of the equation for small axisymmetric steady-state vibrations of an elastic conical shell of revolution. Izv. Akad. Nauk EstSSR, Ser. Tekhn. i fiz.-matem. nauk. Vol. 9, No. 1, 1960.

5. Tovstik, P.E. Integrals of the system of equations for nonaxisymmetric vibrations of shells of revolution. In a Collection: Investigations on Elasticity and Plasticity. (pp.45-55). Izd. LGU,(Leningrad State University)No.5, 1966.
6. Lidskii, V.B. and Khar'kova, N.V. Spectrum of a system of membrane equations in the case of axisymmetric vibrations of shells of revolution. Dokl. Akad. Nauk. SSSR Vol.194, No.4, 1970.
7. Dunford, N. and Schwartz, J.T. Linear Operators. Interscience Publishers New York 1958.
8. Vishik, M.I. and Liusternik, L.A. Regular degeneracy and boundary layer for linear differential equations with a small parameter. J. Usp. Matem. Nauk Vol.12, No.5(77), 1957.
9. Khar'kova, N.V. On the lower portion of the spectrum of natural axisymmetric vibrations of a thin elastic shell of revolution. PMM Vol.35, No.3, 1971.
10. Gokhberg, I.Ts. and Krein, M.G. Fundamental statements on defective numbers, radical numbers and indices of linear operators. J. Usp. Matem. Nauk Vol.12, No.2(74), 1957.
11. Riesz, F. and Sz.-Nagy, B. Functional Analysis, Frederick Ungar, New York, 1955.

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CONSERVATION OF INTEGRALS OF MOTION FOR SMALL CHANGES OF HAMILTON'S FUNCTION IN SOME CASES OF INTEGRABILITY OF THE EQUATIONS OF MOTION OF A GYROSTAT

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Motion of a gyrostat is considered. The equations of motion are written in the Hamilton form and the change in the integrals of motion in the cases of Zhukovskii and Lagrange resulting from the Hamilton function undergoing small variations is studied.

Let the mechanical system under investigation depend on a set of parameters and let it be integrable for some definite values of these parameters. Study of the motion of this system in the case when the values of the parameters are changed the system is no longer integrable, appears to be of interest. The solution of this problem involves overcoming certain fundamental difficulties connected with the problem of small denominators. In the case when the system is Hamiltonian and the changes in the values of parameters are small, these difficulties have been overcome using the method proposed by Kolmogorov and Arnol'd in [1 and 2].

Arnol'd's solution [3] of the problem of a rapidly rotating, heavy, asymmetric rigid body with a fixed point, serves to illustrate the application of this method to the rigid body dynamics.